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ON *CAS*-SUBGROUPS OF FINITE GROUPS*

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ABSTRACT

We introduce a new subgroup embedding property of a finite group called CAS-subgroup. Using this subgroup property, we determine the structure of finite groups with some CAS-subgroups of Sylow subgroups. Our results unify and generalize some recent theorems on solvability, p-nilpotency and supersolvability of finite groups.

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1. Introduction

In 1962, Gaschütz [4] introduced a certain conjugacy class of subgroups of a finite solvable group G with the cover-avoidance property. These subgroups have the property that they avoid the complemented chief factors of G and cover the rest. Thereafter, many authors, Gillam [5] and Tomkinson [10], for example, devoted themselves to find some kind of subgroups of a finite solvable group having this cover-avoidance property. In 1993, Ezquerro [3] considered the converse questions, he gave some characterizations for a finite group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some Sylow subgroup of G are CAP-subgroups. Asaad in 1998 obtained further results in the formation universe [2]. In recent years, it is of interest to use some supplementation properties of subgroups to determine the structure of the given group. For example, Wang introduced the c-normality of a finite group in 1996 which is a kind of supplementation property [11]. By using the c-normality of some maximal and minimal subgroups, he gave some new criteria for the solvability and the supersolvability of finite groups. As a generalization of c-normality and complementation, Wang introduced the c-supplementation of a finite group in 2000 by replacing the normal supplement with the more general supplement [12]. As applications, Wang presented some conditions for a finite group to be solvable, p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are c-supplemented. By limiting the Sylow subgroups to the Fitting subgroup of some solvable group, Wang, Wei and Li extended the results further to saturated formations [13]. Ramadan, Mohamed, Heliel [9] received some good results by assuming that some subgroups of prime power order are c-normal.

In Section 2, we will show that it is easy to find groups with CAP-subgroups which are not c-supplemented subgroups. Conversely, there are also groups with c-supplemented subgroups which are not CAP-subgroups. Two examples in Section 2 show that there is no obvious general relationship between these two notions. Hence a natural question to ask is:

Whether the two concepts and the related results can be unified and generalized?

The purpose of this paper is to present an answer to the above question. In detail, we introduce a new subgroup embedding property of a finite group called CAS-subgroup which is a generalization of the cover-avoidance property and the c-supplementation property. By using this subgroup property, we determine the structure of finite groups with some CAS-subgroups of Sylow subgroups. Our results unify and generalize the related results mentioned above on solvability,

p-nilpotency and supersolvability of finite groups.

2. Preliminaries and basic properties

All groups considered will be finite. For a group G, we denote by $\pi(G)$ the set of prime divisors of $|G|$.

Let G be a group, H a subgroup of G and A/B be a G-chief factor. We say that (i) H covers A/B if $HA = HB$; (ii) H avoids A/B if $H \cap A = H \cap B$. H is called a CAP-subgroup of G if H either covers or avoids any G-chief factor. We mention that H is c-supplemented (c-normal) in G if there exists a (normal) subgroup K of G such that $G = HK$ and $H \cap K \leq H_G = Core_G(H)$ ([11] and [12]). In this case, if we let $K_1 = H_G K$, then $G = HK_1$ with $H \cap K_1 = H_G$; $H \cap K_1$ is, of course, a CAP-subgroup of G. Based on the observation, we introduce the following:

Definition 2.1: Let H be a subgroup of a group G. H is said to be a CAS-subgroup of G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is a CAP-subgroup of G. In this case, K is called a CAS-supplement of H in G.

Remark 2.2: Clearly a CAP-subgroup or c-supplemented subgroup must be a CAS-subgroup. But the converse is not true in general.

Example 1: Let $G = A_5$ be the alternating group of degree five. Then $G = AB$ with $A \cong A_4$ and $B \cong C_5$. We see that both A and B are complemented and so c-supplemented in G but neither of them is a CAP-subgroup of G.

Example 2: We may also find a solvable example which is CAS but is not CAP in G. Let A_4 be the alternative group of degree 4 and $D = \langle d \rangle$ be a cyclic group of order 2. Let $G = D \times A_4$. Then $A_4 = [K_4]C_3$ where $K_4 = \langle a, b \rangle$ is the Klein Four Group with generators a and b of order 2 and C_3 is the cyclic group of order 3. Take $H = \langle ad \rangle$ to be the cyclic subgroup of order 2 of G. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, H is CAS in G. However, H is not a CAP-subgroup of G as it neither covers nor avoids $(D \times K_4)/D$.

Example 3: Let A be a cyclic group of order 5 and let B be the full automorphism group of A of order 4. Let $G = [A]B$. Then the maximal subgroup of B is a CAP-subgroup of G. However it is not c-supplemented in G. In fact, let $B = \langle \alpha \rangle$ with $|\alpha| = 4$. Since $B = \langle \alpha^2 \rangle$ is not normal in G (otherwise $\alpha^2 = 1$), we have that $\langle \alpha^2 \rangle_G = 1$. If $\langle \alpha^2 \rangle$ is c-supplemented in G, then it is complemented in B, which is contrary to $\langle \alpha^2 \rangle$ being the only maximal subgroup of B.

Let G be the direct product of the groups in Example 1 and Example 3. We know that a CAS-subgroup is not necessarily a CAP-subgroup nor c-supplemented.

LEMMA 2.3: *Let H be a subgroup of a group G. Then*

(1) *let $N \lhd G$ and $N \leq H$. If H is a CAS-subgroup of G, then H/N is a CAS-subgroup of G/N;*
(2) *let π be a set of primes, H a π-subgroup and N a normal π'-subgroup of G. If H is a CAS-subgroup of G, then HN/N is a CAS-subgroup of G/N;*
(3) *let $L \leq G$ and $H \leq \Phi(L)$. If H is a CAS-subgroup of G, then H is a CAP-subgroup of G.*

Proof: (1) There is a subgroup K of G such that $G = HK$ and $H \cap K$ is a CAP-subgroup of G. Then $G/N = (H/N)(KN/N)$ and $H/N \cap KN/N = (H \cap K)N/N$ is a CAP-subgroup of G/N. So H/N is a CAS-subgroup of G/N.

(2) Let K be a CAS-supplement of H in G. Then $H \cap K$ is a CAP-subgroup of G. Clearly $N \leq K$, hence $G/N = (HN/N)(K/N)$ and $HN/N \cap K/N = (H \cap K)N/N$ is a CAS-subgroup of G/N. This proves that HN/N is a CAS-subgroup of G/N.

(3) There is a subgroup K of G such that $G = HK$ and $H \cap K$ is a CAP-subgroup of G. It follows from $L = H(L \cap K)$ and $H \leq \Phi(L)$ that $L \cap K = L$. Furthermore, $G = K$ and $H = H \cap K$ is a CAP-subgroup of G. ∎

LEMMA 2.4: *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then G is p-nilpotent:*

(1) *$(|G|, p-1) = 1$ and $|P| \leq p$;*
(2) *G is A_4-free if $p = min\ \pi(G)$;*
(3) *$(|G|, p^2 - 1) = 1$.*

Proof: Let G be a minimal counterexample. For any proper subgroup M of G, we see easily that M satisfies the hypotheses of lemma. It follows that G is a minimal non-p-nilpotent group (that is, every proper subgroup of a group is p-nilpotent but itself is not p-nilpotent). By results of Itô (see, [8, IV, 5.4]) and Schmidt (see, [8, III, 5.2]), G has a normal Sylow p-subgroup G_p and a cyclic Sylow q-subgroup G_q such that $G = [G_p]G_q$. Thus $G_p \cap H = P \lhd G$ and G/P is p-nilpotent. Consequently $P \neq 1$ and $PG_q \lhd G$. If $PG_q < G$ then PG_q is nilpotent and, G_q char $PG_q \lhd G$ implies that $G_q \lhd G$, a contradiction. Therefore

$PG_q = G$ and $P = G_p$. Clearly, $P \leq C_G(P) < G$, so q divides $|G/C_G(P)|$. For (1), $|Aut(P)| = p - 1$; for (2) and (3), $|P| = p^2$ and $|Aut(P)| = (p^2 - 1)(p^2 - p)$. Since $G/C_G(P)$ is isomorphic to a subgroup of $Aut(P)$, q must divide $p - 1$ or $p^2 - 1$. This is contrary to (1) and (3). If (2) is satisfied, then $q|p + 1$ and so $p = 2$ and $q = 3$. It is now clear that $G/\Phi(G_q)$ is isomorphic to A_4, which is contrary to the hypothesis that G is A_4-free. ∎

THEOREM 2.5: *Let G be a finite group. Then G is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are CAS-subgroups of G.*

Proof: If G is solvable, then, by [8, main theorem], every Sylow subgroup of G is complemented. In particular, every Sylow 2-subgroup and every Sylow 3-subgroup are CAS-subgroups of G.

Conversely, assume that every Sylow 2-subgroup and every Sylow 3-subgroup are CAS-subgroups of G. We proceed to prove that G is solvable. Suppose not, and let G be a minimal counterexample. Then

(1) G has a unique minimal normal subgroup N such that G/N is solvable.

Let N be a minimal normal subgroup of G. We shall show that $\overline{G} = G/N$ satisfies the hypotheses of the theorem. For this purpose, let $\overline{P} = PN/N$ be a Sylow 2-subgroup of $\overline{G}$, where P is a Sylow 2-subgroup of G. By the hypotheses, there exists a subgroup K of G such that $G = PK$ and $P \cap K$ is a CAP-subgroup of G. Now let $\pi(G) = \{p_1, p_2, \ldots, p_n\}$ and let K_{p_i} be a Sylow p_i-subgroup of K, where $p_1 = 2$ and $i = 2, \ldots, n$. Then K_{p_i} is also a Sylow p_i-subgroup of G, so $N \cap K_{p_i}$ is a Sylow p_i-subgroup of N. If we denote $L = \langle N \cap K_{p_2}, \ldots, N \cap K_{p_n} \rangle$, then $L \leq K$ and $N = (P \cap N)L$. It follows that $PN \cap KN = (P \cap K)N$ is a CAP-subgroup of G, i.e., PN is a CAS-subgroup of G. By Lemma 2.3, $\overline{P}$ is a CAS-subgroup of $\overline{G}$. Similarly, every Sylow 3-subgroup of $\overline{G}$ is also a CAS-subgroup of $\overline{G}$. Thus $\overline{G}$ satisfies the hypotheses of the theorem. The choice of G implies that $\overline{G}$ is solvable. Since the class of solvable groups is a formation, N is the unique minimal normal subgroup of G.

(2) Every Sylow 2-subgroup of G is complemented in G.

Let P be a Sylow 2-subgroup of G and let K be a CAS-supplement of P in G. Then $G = PK$ and $P \cap K$ is a CAP-subgroup of G. If $P \cap K$ covers $N/1$, then $(P \cap K)N = P \cap K$, $N \leq P \cap K$. In this case, G is solvable since both N and G/N are, a contradiction. Hence $P \cap K$ must avoid $N/1$, namely $P \cap K \cap N = 1$. It follows that $K \cap N$ is a $2'$-group. By the Odd Order Theorem, $K \cap N$ is solvable. Now that $K/K \cap N \cong KN/N \leq G/N$, $K/K \cap N$ is solvable and so is K. It is clear that a 2-complement of K is also a 2-complement of G. This proves (2).

(3) Every Sylow 3-subgroup of G is complemented in G.

Let Q be a Sylow 3-subgroup of G. By hypotheses, there exists a subgroup H of G such that $G = QH$ and $Q \cap H$ is a CAP-subgroup of G. Clearly, $Q \cap H$ cannot cover $N/1$, so $Q \cap H \cap N = 1$, namely $H \cap N$ is a $3'$-group. If we take P to be a Sylow 2-subgroup of H, then P is also a Sylow 2-subgroup of G. By (2), P has a complement in G, say K. We have $H = P(H \cap K)$. Again, $H \cap N \lhd H$, so $P \cap N$ is a Sylow 2-subgroup complementing $H \cap K \cap N$. This shows that every Sylow 2-subgroup of the $3'$-group $H \cap N$ is complemented in $H \cap N$. By Arad–Ward's theorem [1], $H \cap N$ is solvable. Now $H/H \cap N \cong HN/N \leq G/N$ is solvable, hence H is also solvable. Now, the 3-complements of H are also 3-complements of G, thus Q is complemented in G.

Finally, we deduce that G is solvable by Arad–Ward's theorem [1], a contradiction. ∎

Corollary 2.6: *A finite group G is solvable if and only if every Sylow subgroup of G is a CAS-subgroup of G.*

3. CAS-subgroups and the p-nilpotency

In this section, $\mathcal{N}_p$ and $G^{\mathcal{N}_p}$ will denote the class of p-nilpotent groups and the p-nilpotent residual of G, respectively.

Theorem 3.1: *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are CAS-subgroups of G, then G is p-nilpotent.*

Proof: Let G be a minimal counterexample. Then

(1) G has a unique minimal normal subgroup N contained in H, G/N is p-nilpotent and $N \not\leq \Phi(G)$.

Let N be a minimal normal subgroup of G contained in H. Consider the factor group $\overline{G} = G/N$. Clearly $\overline{G}/\overline{H} \cong G/H$ is p-nilpotent and $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{H}$, where $\overline{H} = H/N$. Now let $\overline{P_1} = P_1N/N$ be a maximal subgroup of $\overline{P}$. We may assume that P_1 is a maximal subgroup of P. Then $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. By hypotheses, there exists a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. We have $G/N = (P_1N/N)(K_1N/N)$ and $P_1N \cap K_1N = (P_1N \cap K_1)N$.

Let $\pi(G) = \{p_1, p_2, \ldots, p_n\}$ and let K_{1p_i} be a Sylow p_i-subgroup of K_1, where $p_1 = p$ and $i = 2, \ldots, n$. Then K_{1p_i} is also a Sylow p_i-subgroup of G, hence $N \cap K_{1p_i}$ is a Sylow p_i-subgroup of N. Write $L = \langle N \cap K_{1p_2}, \ldots, N \cap K_{1p_n} \rangle$. Then $L \leq K_1$. Note that $(|N : P_1 \cap N|, |N : L|) = 1$, $N = (P_1 \cap N)L$ and $P_1N/N \cap K_1N/N = (P_1 \cap K_1)N/N$ is a CAP-subgroup of G/N. Thus $\overline{P}_1$ is a CAS-subgroup of $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation [6, VI, 7.6], N is the unique minimal normal subgroup of G contained in H and $N \not\leq \Phi(G)$.

(2) $O_p(H) = 1$.

If not, then by (1), $N \leq O_p(H)$ and, there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. It follows that $M \cong G/N$ is p-nilpotent. Let $M_{p'}$ be the normal p-complement of M, then $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that either $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we must have $N_G(M_{p'}) = M$. On the other hand, since $P \cap M < P$, we may let P_1 be a maximal subgroup of P containing $P \cap M$. Because P_1 is a CAS-subgroup of G, there exists a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. If $P_1 \cap K_1$ covers $N/1$, then $(P_1 \cap K_1)N = P_1 \cap K_1$, that is, $N \leq P_1 \cap K_1$. Furthermore, $P = N(P \cap M) \leq P_1$, a contradiction. Thereby $P_1 \cap K_1$ must avoid $N/1$, i.e., $P_1 \cap K_1 \cap N = 1$. Consequently, $|K_1 \cap N| \leq p$. Since $K_1/N \cap K_1 \cong K_1N/N \leq G/N$, $K_1/N \cap K_1$ is p-nilpotent. It follows that K_1 is p-nilpotent by Lemma 2.4. Now let $K_{1p'}$ be the normal p-complement of K_1. Then $K_{1p'} \lhd K_1$. Clearly, both $K_{1p'}$ and $M_{p'}$ are Hall p'-subgroups of G of odd order. By applying a deep result of Gross ([6, main Theorem]), there exists $g \in G$ such that $K_{1p'}^g = M_{p'}$. Hence $K_1^g \leq N_G(K_{1p'}^g) = N_G(M_{p'}) = M$. However, $K_{1p'}$ is normalized by K_1, so g can be considered as an element of P_1. Thus $G = P_1K_1^g = P_1M$ and $P = P_1(P \cap M) = P_1$, a contradiction.

(3) End of the proof.

Let P_1 be a maximal subgroup of P. Then there exists a subgroup K_1 such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. It follows that $H = P_1(H \cap K_1)$ and $P \cap K_1$ is a Sylow p-subgroup of $H \cap K_1$. Again, $K_1 \cap N \lhd H \cap K_1$, hence $P \cap K_1 \cap N$ is a Sylow p-subgroup of $K_1 \cap N$. By (2), we see that $N \not\leq P_1 \cap K_1$, so $P_1 \cap K_1 \cap N = 1$ and $|P \cap K_1 \cap N| \leq p$. However, $K_1/K_1 \cap N \cong K_1N/N \leq G/N$, so $K_1/K_1 \cap N$ is p-nilpotent and, K_1 is p-nilpotent by Lemma 2.4. Let $K_{1p'}$ be the normal p-complement of K_1 and let $R = N_G(K_{1p'})$. Then $K_1 \leq R$ and $P = P_1(P \cap R)$. If $R = G$, $K_{1p'} \lhd G$,

$K_{1p'}$ is actually the normal p-complement of G, a contradiction. Thus $R < G$ and $P \cap R < P$. Now take P_2 to be a maximal subgroup of P containing $P \cap R$ and K_2 to be a CAS-supplement of P_2 in G. Then $P_2 \cap K_2$ is a CAP-subgroup of G. Similarly, K_2 is also p-nilpotent. If $K_{2p'}$ is the normal p-complement of K_2, then $K_2 \leq N_G(K_{2p'})$. Since both $K_{2p'}$ and $K_{1p'}$ are Hall p'-subgroups of G of odd order, by Gross' Theorem, there is $g \in G$ such that $K_{2p'}^g = K_{1p'}$. Since $K_{2p'}$ is normalized by K_2, g can be considered as an element of P_2. Therefore $K_2^g \leq N_G(K_{2p'}^g) = N_G(K_{1p'}) = R$. Now we obtain that $G = P_2K_2^g = P_2R$ and $P = P_2(P \cap R) = P_2$, a contradiction. We are done. ∎

Corollary 3.2: *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If G is not p-nilpotent, then there is a maximal subgroup of $P \cap G^{\mathcal{N}_p}$ which is not a CAS-subgroup of G.*

Theorem 3.3: *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P a Sylow p-subgroup of H, where p is the smallest prime divisor of $|G|$. If G is A_4-free and every second maximal subgroups of P is a CAS-subgroup of G, then G is p-nilpotent.*

Proof: Let G be a minimal counterexample. Then

(1) G has a unique minimal normal subgroup N contained in H, G/N is p-nilpotent and $N \not\leq \Phi(G)$.

(2) $O_p(H) = 1$.

If not, then $N \leq O_p(H)$ by (1). Moreover, there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Therefore $M \cong G/N$ is p-nilpotent. Let $M_{p'}$ be the normal p-complement of M. Then $M \leq N_G(M_{p'}) \leq G$. The maximality of M in G implies that either $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we must have $N_G(M_{p'}) = M$. In this case, if $N = P$, then $|N| > p^2$ by Lemma 2.4. Take a second maximal subgroup P_1 of N. Since P_1 is a CAS-subgroup of G, there exists a subgroup K_1 such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. Clearly $P_1 \cap K_1$ must avoid $N/1$, therefore $P_1 \cap K_1 = 1$. It follows from $K_1/N \cap K_1 \cong K_1N/N \leq G/N$ that $K_1/N \cap K_1$ is p-nilpotent. However, $|N \cap K_1| \leq p^2$, so K_1 is p-nilpotent by Lemma 2.4. Now let $K_{1p'}$ be the normal p-complement of K_1. We see that $K_{1p'} \lhd K_1$. Clearly, both $K_{1p'}$ and $M_{p'}$ are Hall p'-subgroups of G of odd order. By Gross' result, there is $x \in G$ such that $K_{1p'}^x = M_{p'}$. Hence $K_1^x \leq N_G(K_{1p'}^x) = N_G(M_{p'}) = M$. Now that $K_{1p'}$ is normalized by K_1, x can be considered as an element of P_1. Thus $G = P_1K_1^x = P_1M$ and $P = P_1(N \cap M) = P_1$, a contradiction.

Now we assume that $N < P$. Clearly $P \cap M < P$, so we may let P_2 be a maximal subgroup of P such that $P \cap M \leq P_2$. Then $P_2 = (P_2 \cap N)(P \cap M)$. Furthermore,

$$|N : P_2 \cap N| = |N(P \cap M) : (P_2 \cap N)(P \cap M)| = |P : P_2| = p,$$

that is, $P_2 \cap N$ is a maximal subgroup of N. Note that $P \cap M \neq 1$, let P_3 be a maximal subgroup of $P \cap M$. Since $P_2 \cap N \lhd P$ and $|P : (P_2 \cap N)P_3| = p^2$, $P_0 = (P_2 \cap N)P_3$ is a second maximal subgroup of P. Thus, by hypotheses, there is a subgroup K_0 such that $G = P_0K_0$ and $P_0 \cap K_0$ is a CAP-subgroup of G. If $P_0 \cap K_0$ covers $N/1$, then $N \leq P_0$ and $N \leq P_0 \cap N = P_2 \cap N$, contrary to the maximality of $P_2 \cap N$ in N. So $P_0 \cap K_0$ must avoid $N/1$, namely $P_0 \cap K_0 \cap N = 1$. Furthermore, $|K_0 \cap N| \leq p^2$. Again, $K_0/K_0 \cap N \cong K_0N/N \leq G/N$ is p-nilpotent, hence K_0 is p-nilpotent by Lemma 2.4. Let $K_{0p'}$ be the normal p-complement of K_0. Clearly both $K_{0p'}$ and $M_{p'}$ are Hall p'-subgroups of G, again by Gross' result, there is $y \in G$ such that $K_{0p'}^y = M_{p'}$. Hence $K_0^y \leq N_G(K_{0p'}^y) = N_G(M_{p'}) = M$. Similarly, y can be regarded as an element of P_0. Thus $G = P_0K_0^y = P_0M = (P_2 \cap N)M$ and $N = P_2 \cap N$, a contradiction. This proves (2).

(3) A final contradiction.

By Lemma 2.4, we may assume that $|N|_p > p^2$. Let P_1 be a second maximal subgroup of P. Then there exists a subgroup K_1 such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. We have $H = P_1(H \cap K_1)$. It is easy to see that $P \cap K_1 \cap N$ is a Sylow p-subgroup of $K_1 \cap N$. By (2), $N \nleq P_1 \cap K_1$, so $P_1 \cap K_1 \cap N = 1$ and $|K_1 \cap N|_p \leq p^2$. From $K_1/K_1 \cap N \cong K_1N/N \leq G/N$ we see that $K_1/K_1 \cap N$ is p-nilpotent. Therefore K_1 is p-nilpotent by Lemma 2.4. Let $K_{1p'}$ be the normal p-complement of K_1 and let $R = N_G(K_{1p'})$. We may assume that $R < G$. Take P_2 to be a maximal subgroup of P containing $P \cap R$. Then $P_2 \cap R = P \cap R$ and $P_2 = (P_1 \cap P_2)(P \cap R)$. Hence

$$|P_1 : P_1 \cap P_2| = |P_1(P \cap R) : (P_1 \cap P_2)(P \cap R)| = |P : P_2| = p,$$

that is, $P_1 \cap P_2$ is a maximal subgroup of P_1. Again, $P_1 \cap R < P \cap R$, so we may choose P_3 to be a maximal subgroup of $P \cap R$ containing $P_1 \cap R$. Furthermore, we get $P_1 \cap P_3 = P_1 \cap R$. Since $P_1 \cap P_2 \lhd P$, $(P_1 \cap P_2)P_3$ is a group and,

$$|P : (P_1 \cap P_2)P_3| = |P_1(P \cap R) : (P_1 \cap P_2)P_3| = p^2.$$

Thus $P_0 = (P_1 \cap P_2)P_3$ is a second maximal subgroup of P. Let K_0 be a CAS-supplement of P_0 in G. With similar arguments, we see that K_0 is also p-nilpotent. Assume that $K_{0p'}$ is the normal p-complement of K_0. Now both $K_{0p'}$ and $K_{1p'}$ are Hall p'-subgroups of G of odd order, so there exists $x \in G$ such that $K_{0p'}^x = K_{1p'}$. We obtain $K_0^x \leq N_G(K_{0p'}^x) = N_G(K_{1p'}) = R$. Of course, x can be considered as an element of P_0. Thus $G = P_0K_0^x = P_0R = (P_1 \cap P_2)R$ and $P_1 = (P_1 \cap P_2)(P_1 \cap R)$. Now from $P_1 \cap R \leq P_1 \cap P_2$ we have $P_1 = P_1 \cap P_2$, which is contrary to the maximality of $P_1 \cap P_2$ in P_1. ∎

Corollary 3.4: *Let G be a group which is A_4-free and let P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If G is not p-nilpotent, then there is a maximal subgroup of $P \cap G^{\mathcal{N}_p}$ which is not a CAS-subgroup of G.*

Similarly, we have the following:

Theorem 3.5: *Let H be a subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime number dividing $|G|$ with $(|G|, p^2-1) = 1$. If every second maximal subgroup of P is a CAS-subgroup of G, then G is p-nilpotent.*

4. CAS-subgroups and the supersolvability

For convenience, we denote by $\mathcal{U}$ and $G^{\mathcal{U}}$ the class of supersolvable groups and the supersolvable residual of G, respectively.

Lemma 4.1 ([13, Lemma 2.8]): *Let M be a maximal subgroup of G, P a normal p-subgroup of G such that $G = PM$, where p a prime. Then*

(1) $P \cap M$ is a normal subgroup of G;

(2) if $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.

Lemma 4.2 ([13, Theorem 3.1]): *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$.*

Theorem 4.3: *Suppose that H is a normal subgroup of a group G such that G/H is supersolvable. If all maximal subgroups of any Sylow subgroup of H are CAS-subgroups of G, then G is supersolvable.*

Proof: Let G be a minimal counterexample. Then

(1) G has a unique minimal normal subgroup N contained in H, G/N is supersolvable and $N \nleq \Phi(G)$.

(2) N is an elementary abelian p-group but is not a Sylow p-subgroup of H, where $p \in \pi(H)$.

Let r be the smallest prime divisor of $|G|$. By the hypothesis and Theorem 3.1, G is r-nilpotent. In particular, G is solvable by the Odd Order Theorem. Hence N is an elementary abelian p-group for some $p \in \pi(H)$. Assume that N is a Sylow subgroup of H. Take a maximal subgroup P_1 of N. By hypothesis, there is a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. Clearly, $P_1 \cap K_1$ cannot cover $N/1$, so $(P_1 \cap K_1) \cap N = 1$, that is, $P_1 \cap K_1 = 1$. On the other hand, $N \cap K_1 \lhd G$, so $N \cap K_1 = 1$ or N by the minimal normality of N. If $N \cap K_1 = 1$ then $N = P_1$, in a contradiction. Hence $N \cap K_1 = N$ and $|N| = p$. By Lemma 4.2, G is supersolvable, also a contradiction. (2) follows.

(3) $p \neq q$, where q is the largest prime divisor of $|G|$.

Assume that $p = q$. By (1) and (2), there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Let M_p be a Sylow p-subgroup of M. Then NM_p is a Sylow p-subgroup of G and so it is normal in G because G/N is supersolvable. Write $H_p = NM_p \cap H$. Then H_p is a normal subgroup of G. In this case, by Lemma 4.1, $H_p \cap M$ is normal in G. If further $H_p \cap M \neq 1$, then $N \leq H_p \cap M$ by (1), which is contrary to $N \cap M = 1$. Hence we must have $H_p \cap M = 1$ and consequently, $N = H_p$ is a Sylow p-subgroup of H, contrary to (2).

(4) There is a maximal subgroup M of G such that $N_G(Q) = M$, where $Q \in \mathrm{Syl}_q(M)$.

From (3) we see that there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. It follows that $M \cong G/N$ is supersolvable. Let Q be a Sylow q-subgroup of M. Then Q is normal in M and $M \leq N_G(Q) \leq G$. The maximality of M implies that either $M = N_G(Q)$ or $N_G(Q) = G$. If $N_G(Q) = G$, then $Q \lhd G$ and, by (1) and (3), we obtain $Q \cap H = 1$. However, Q is also a Sylow q-subgroup of G, so H is a q'-group. Now consider the factor group $\overline{G} = G/Q$. Clearly, $\overline{G}/\overline{H} \cong G/HQ$ is supersolvable, where $\overline{H} = HQ/Q$. Let $\overline{R}_1 = R_1Q/Q$ be a maximal subgroup of some Sylow r-subgroup of $\overline{H}$, where $r \in \pi(H)$ and $r \neq q$. We may assume that R_1 is a maximal subgroup of some Sylow r-subgroup R of H. By the hypothesis, R_1 is a CAS-subgroup of G, so R_1Q/Q is a CAS-subgroup of G/Q by Lemma 2.3. This means that $\overline{G}$ satisfies the hypothesis of theorem. By the choice of G, G/Q is supersolvable. Again by (1), we have that $G \cong G/(N \cap Q)$ is supersolvable, a contradiction. Thus $N_G(Q) = M$ and (4) follows.

(5) End of the proof.

Let P be a Sylow p-subgroup of H. Now that $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Now that P_1 is a CAS-subgroup of G, there is a subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. If $P_1 \cap K_1$ covers $N/1$, then $(P_1 \cap K_1)N = P_1 \cap K_1$, that is, $N \leq P_1 \cap K_1$, thus $P = N(P \cap M) \leq P_1$, a contradiction. Thereby $P_1 \cap K_1$ must avoid $N/1$, namely $P_1 \cap K_1 \cap N = 1$. Furthermore, $|K_1 \cap N| \leq p$. Again, $K_1/K_1 \cap N \cong K_1N/N \leq G/N$, so $K_1/K_1 \cap N$ is supersolvable. In this case, K_1 is supersolvable by Lemma 4.2. Let K_{1q} be a Sylow q-subgroup of K_1. Then K_{1q} is a normal subgroup of K_1 and a Sylow q-subgroup of G. On the other hand, Q is also a Sylow q-subgroup of G, so by Sylow theorem, there is $g \in G$ such that $K_{1q}^g = Q$. Hence $K_1^g \leq N_G(K_{1q}^g) = N_G(Q) \leq M$. Moreover, g can be considered as an element of P_1, hence $G = P_1K_1^g = P_1M$ and $P = P_1(P \cap M) = P_1$, a contradiction. ∎

Since every subgroup of a supersolvable group must be a CAP-subgroup, we have the following:

Corollary 4.4: *A group G is supersolvable if and only if every maximal subgroup of any Sylow subgroup of G is a CAS-subgroup of G.*

Remark 4.5: There exists a supersolvable group such that some maximal subgroup of Sylow subgroup is not c-supplemented in G. Example 2 in Remark 2.2 is a counterexample.

Corollary 4.6: *If G is not supersolvable, then there is a maximal subgroup of Sylow subgroup of $G^{\mathcal{U}}$ which is not a CAS-subgroup of G.*

([12, Theorem 3.3]) follows directly from Corollary 4.6.

Theorem 4.7: *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let H be a solvable normal subgroup of G such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F(H)$ are CAS-subgroups of G, then $G \in \mathcal{F}$.*

Proof: Let G be a minimal counterexample. First we have $\Phi(H) = 1$. In fact, if $\Phi(H) \neq 1$, we consider $\overline{G} = G/\Phi(H)$. Obviously, $F(\overline{H}) = F(H)/\Phi(H)$. Now it is not hard to show that $\overline{G}$ satisfies the hypothesis of theorem. By the minimality of G we obtain $G/\Phi(H) \in \mathcal{F}$. However, $\Phi(H) \leq \Phi(G)$, hence $G \in \mathcal{F}$

since $\mathcal{F}$ is saturated, a contradiction. Let M be a maximal subgroup of G such that $F(H) \nleq M$. Then there exists a prime p such that $O_p(H) \nleq M$. It follows that $G = O_p(H)M$. Clearly, $O_p(H) \cap M < O_p(H)$, so we may take a maximal subgroup P_1 of $O_p(H)$ containing $O_p(H) \cap M$. Then $P_1 \cap M = O_p(H) \cap M \lhd G$, therefore $P_1 \cap M \leq (P_1)_G$. Again, $(P_1)_G M < G$, so $(P_1)_G \leq O_p(H) \cap M$ and $P_1 \cap M = O_p(H) \cap M = (P_1)_G$. Let $O_p(H)/K$ be a chief factor of G with $O_p(H) \cap M \leq K$. Then $O_p(H) \cap M = K \cap M$. Now, $KM < G$, so $K \leq O_p(H) \cap M$ and $K = O_p(H) \cap M = (P_1)_G$. Since $P_1 \cap K_1$ is a CAS-subgroup of G, there exists a subgroup K_1 such that $G = P_1 K_1$ and $P_1 \cap K_1$ is a CAP-subgroup of G. Clearly $(P_1)_G(O_p(H) \cap K_1)$ is normal in G. From the fact that $O_p(H)/(P_1)_G$ is a G-chief factor we know that either $(P_1)_G = (P_1)_G(O_p(H) \cap K_1)$ or $(P_1)_G(O_p(H) \cap K_1) = O_p(H)$. If the former holds, then $O_p(H) \cap K_1 \leq (P_1)_G$. Furthermore, $O_p(H) \cap K_1 = P_1 \cap K_1$ and $O_p(H) = P_1$ as $P_1 K_1 = O_p(H) K_1 = G$, a contradiction. So $(P_1)_G(O_p(H) \cap K_1) = O_p(H)$, we obtain $O_p(H) \leq (P_1)_G K_1$. Thus $G = (P_1)_G K_1 = P_1 K_1$. But $(P_1)_G \cap K_1 = P_1 \cap K_1$, we have $P_1 = (P_1)_G = O_p(H) \cap M$. Therefore $|G : M| = |O_p(H) : O_p(H) \cap M| = p$. By Lemma 4.2, we get $G \in \mathcal{F}$, a final contradiction. ∎

([13, Theorem 4.2]) and ([2, Theorem 4.3]) follow directly from Theorem 4.7.

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